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Mixing quantum and classical mechanics and uniqueness of Planck's constant

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Abstract
Observables of quantum or classical mechanics form algebras called quantum or classical Hamilton algebras (HAs), respectively (Grgin and Petersen 1974 *J. Math. Phys.* **15** 764, Sahoo 1977 *Pramana* **8** 545). We show that the tensor product of two quantum HAs, each characterized by a different Planck's constant (PC), is an algebra of the same type characterized by yet another PC. The algebraic structure of mixed quantum and classical systems is then analysed by taking the limit of vanishing PC in one of the component algebras. This approach provides new insight into failures of various formalisms dealing with mixed quantum–classical systems. It shows that in the interacting mixed quantum–classical description, there can be no back-reaction of the quantum system on the classical system. A natural algebraic requirement involving restriction of the tensor product of two quantum HAs to their components proves that PC is unique.

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1. Introduction

The title of the present paper may create the impression that two disjoint subjects are being discussed together. However, a little reflection would convince the reader that there is a connecting thread between the two. Inherent in a quantum system is a Planck's constant (PC) governing its behaviour whereas a classical system can be thought of as a system with zero PC. Thus in mixed quantum–classical mechanics, we are dealing with two systems with different PCs. Now in proving the uniqueness of PC, it is but natural to consider two systems with different values of PC and then to examine the consequences. The fact that two different PCs come into play in the analysis of both the subjects, provides the

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connecting thread. The purpose of this work is two-fold: to investigate *why one cannot have a fundamentally satisfactory dynamical description of interacting quantum–classical systems* and *to understand the uniqueness of PC* conventionally assumed in the physics literature. Our method investigation is of an algebraic nature.

As of now there is no consistent theory of interaction of a classical system with a quantum one. Such a theory is desirable since a variety of problems in a number of different fields involve coupling of quantum and classical degrees of freedom. In the development of quantum mechanics (QM), Bohr [3] always insisted that measuring instruments must be describable in classical terms, but did not provide a theoretical framework for the description of interacting quantum–classical systems. The so-called Copenhagen interpretation of quantum theory, founded on this assumption, is by and large accepted by all physicists; yet this lacuna has remained as a sore point. The issue of a mixed quantum–classical description is important in the discussion of early universe physics where fully quantum matter fields have necessarily to be coupled to the gravitational field which is classical. The traditional approach to this problem has been to couple the gravity field to the expectation values of the quantum energy–momentum tensor of the matter fields. In this approach one misses the effects of quantum fluctuations on the classical gravitational field—the so-called quantum back-reaction.

There has been no dearth of effort in constructing a mathematically consistent theory of such mixed systems. Some authors [5–8] use mixed classical–quantum notation to denote the dynamical variables (DVs) of the (mixed) system. Let $x, y, x_i, \ldots$ denote classical DVs which are ordinary functions of commuting phase-space variables and let $X, Y, X_i, \ldots$ denote quantum DVs which are noncommuting operators (acting on some suitable Hilbert space of states). Then typical DVs of the mixed quantum–classical system are denoted as $Xx, Yy, \ldots$, which are operator-valued functions. Also, the general DVs of the mixed system may be denoted as $\mathcal{X}, \mathcal{Y}, \ldots$ where $\mathcal{X} = \sum_i X_i x_i$, $\mathcal{Y} = \sum_i Y_i y_i$, etc. Let us use the notation

$$[X, Y]^- = \frac{(XY - YX)}{\mathrm{i}\hbar} \tag{1}$$

$$[X, Y]^+ = \tfrac{1}{2}(XY + YX) \tag{2}$$

to denote the commutator and the anticommutator brackets, respectively, and let $\{x, y\}_P$ denote the usual Poisson bracket of classical DVs. In order to denote the corresponding bracket of mixed DVs, let us adopt the notation $[\{,\}]$. Guided mainly by guesswork, the following definitions of this bracket have been proposed:

$$[\{xX, yY\}] = xy[X, Y]^- + \{x, y\}_P [X, Y]^+ \quad [6] \tag{3}$$

$$[\{\mathcal{X}, \mathcal{Y}\}] = [\mathcal{X}, \mathcal{Y}] + \tfrac{1}{2}\left(\{\mathcal{X}, \mathcal{Y}\}_P - \{\mathcal{Y}, \mathcal{X}\}_P\right) \quad [9, 8] \tag{4}$$

$$[\{\mathcal{X}, \mathcal{Y}\}] = [\mathcal{X}, \mathcal{Y}] + \{\mathcal{X}, \mathcal{Y}\}_P \quad [5]. \tag{5}$$

Caro and Salcedo [8] consider a quantum system consisting of two mutually interacting subsystems and enquire whether it is possible to take the classical limit (i.e. letting $\hbar \to 0$) in just one of the subsystems maintaining at the same time an internally consistent dynamics for the resulting mixed quantum–classical system. They call this the *semi-quantization problem* and arrive at a bracket (4). Further, they show that this bracket, although antisymmetric, does *not* satisfy the Jacobi identity. In his investigation of quantum back-reaction on classical variables, Anderson [5] suggests the bracket (5), which is not even antisymmetric. Prezhdo and Kisil [10] develop a mathematically sophisticated formalism and arrive at a result identical to (3) but written in terms of the symbols of the operators (see their equation (24)).

A satisfactory bracket (,) describing any dynamics, be it classical or quantum (and also desirable for a mixed classical–quantum system), must possess the following properties [11]:

$$(A, B) = -(B, A) \quad \text{(antisymmetry)} \tag{6}$$

$$((A, B), C) + ((B, C), A) + ((C, A), B) = 0 \quad \text{(Jacobi identity)} \tag{7}$$

$$(A, BC) = (A, B)C + B(A, C) \quad \text{(derivation identity).} \tag{8}$$

As is well known [8], antisymmetry of the bracket ensures conservation of energy, the Jacobi identity ensures that (A, B) also evolves dynamically and the derivation identity ensures that the product AB also evolves with consistent dynamics. Lack of any of these properties imposes severe impediments to the mixing of classical and quantum degrees of freedom. The bracket (5) has none of these properties whereas the bracket (3) satisfies neither the Jacobi nor the derivation identity [8, 12]. Diosi and his group [13] extensively investigate the question of coupling of quantum–classical systems focusing their attention mainly on maintaining the positivity of the quantum states. Hay and Peres [14] treat the apparatus quantum mechanically while it interacts with the system and then give a classical description of the apparatus within the framework of the Wigner functions. Peres and Temo [15] develop a hybrid formalism by taking recourse to the Koopman operator representation of classical Hamiltonians and conclude that the correspondence principle is violated due to the interaction. Belavkin and his collaborators [16] develop a stochastic Hamiltonian theory for coupling a quantum system with an apparatus. In this approach, attention is focused on providing purely dynamical arguments to derive entanglement, decoherence and collapse of the coupled system consisting of a quantum system and a (semi-classical) apparatus. However, in this theory, as in the original von Neumann theory [17] of measurement, both the system and the apparatus are treated quantum mechanically and a 'reduction model' is proposed achieving some improvement over the von Neumann reduction postulate. Further discussion of this model will be given in the last section dealing with discussion of our results. Sudarshan and his collaborators [18] propose a novel procedure for coupling a classical system (the apparatus) with a quantum one. They embed the classical system into what they term a classical enlarged quantum system (CEQS). The set of observable phase-space variables (regarded as commuting operators) is supplemented by an equal number of unobservable conjugate variables (noncommuting with the previous set). In this approach, after coupling the CEQS with the quantum system and subsequently decoupling these systems, the value of the measured quantity of the quantum system is transferred to an appropriate observable of the CEQS. A certain principle of integrity invoked by these authors assures satisfactory behaviour of observables of the classical system in the measurement process. However, the state evolution of the CEQS indicates that the classical system does *not* remain purely classical after the interaction is over. The basic mathematical problem of a truly satisfactory quantum–classical coupling remains unsolved. The question whether there exists a satisfactory bracket for the mixed quantum–classical systems calls for a detailed investigation from a purely algebraic point of view. The present work has this as one of its main motivations.

The second important result which is derived in this work is an algebraic proof that PC is unique. The possibility of a multiplicity of PCs cannot be logically ruled out and its universality is of an empirically established nature [19]. Fischback *et al* [20] have examined this question carefully and after recalling how the existence of several PCs leads to violation of spacetime symmetry laws, suggest a possible test for experimentally verifying this assumption.

Battaglia [21], while arguing that the introduction of several PCs is undesirable, suggests remedies to some of the embarrassing problems arising in the event that experiments do allow for such an eventuality. Our proof of uniqueness of PC resolves this issue. We now describe the abstract algebraic structures of quantum and classical mechanics.

2. Hamilton algebra

In order to motivate the definition of the algebra of observables of QM we note that in the von Neumann formulation [17] of QM, the associative algebra $\mathcal{B}(\mathcal{S})$ of bounded linear operators, defined over the complex field $\mathbb{C}$, acts on a Hilbert space $\mathcal{S}$ of states. The set of observables $\bar{\mathcal{B}}(\mathcal{S})$, consisting of self-adjoint elements of $\mathcal{B}(\mathcal{S})$ and defined over the real field $\mathbb{R}$, inherits from $\mathcal{B}(\mathcal{S})$ the structure of a Jordan–Lie algebra [22] with a Jordan product $[X, Y]^+$ and a Lie product $[X, Y]^-$ where $X, Y \in \bar{\mathcal{B}}(\mathcal{S})$. Since every $X \in \mathcal{B}(\mathcal{S})$ can be written uniquely in the form $X = X_1 + \mathrm{i}X_2$ with $X_1, X_2 \in \bar{\mathcal{B}}(\mathcal{S})$ and $\mathrm{i} = \sqrt{-1}$, $\mathcal{B}(\mathcal{S})$ is the complex extension of $\bar{\mathcal{B}}(\mathcal{S})$. This observation suggests the definition of a quantum Hamilton algebra (QHA) [2, 4]: it is a two-product algebra $\{\mathcal{H}, \alpha^a, \sigma^a, \mathbb{R}\}$, over the real field $\mathbb{R}$, parametrized by a real number a called the *quantum constant*. Here $\mathcal{H}$ is the linear space underlying the algebra; α^a and σ^a are bilinear products α^a, σ^a: $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$. Henceforth, we shall denote this algebra by the notation $\mathcal{H}^a$. The correspondence between the symbols and their abstract counterparts is $\bar{\mathcal{B}}(\mathcal{S}) \to \mathcal{H}$, $[,]^- \to \alpha^a$, $[,]^+ \to \sigma^a$ and $a \to \hbar^2/4$. Elements of the set $\mathcal{H}$ are denoted by $e, f, g, h, \ldots$, where e is the unit element of $\mathcal{H}^a$ (with respect to σ^a). Note that in the algorithmic form, we have earlier used the notation X, Y, etc to denote the (operator) elements of the elements of $\bar{\mathcal{B}}(\mathcal{S})$. A QHA is defined [2, 4] by the identities:

$$f\alpha^a g = -g\alpha^a f \quad \text{(antisymmetry)} \tag{9}$$

$$f\alpha^a(g\alpha^a h) + g\alpha^a(h\alpha^a f) + h\alpha^a(f\alpha^a g) = 0 \quad \text{(Jacobi identity)} \tag{10}$$

$$f\sigma^a g = g\sigma^a f \quad \text{(symmetry)} \tag{11}$$

$$f\alpha^a(g\sigma^a h) = (f\alpha^a g)\sigma^a h + g\sigma^a(f\alpha^a h) \quad \text{(derivation of } \alpha^a \text{ wrt } \sigma^a \text{)} \tag{12}$$

$$\begin{aligned}\Delta^a_\sigma(f, g, h) &\equiv (f\sigma^a g)\sigma^a h - f\sigma^a(g\sigma^a h)\\ &= a[(f\alpha^a h)\alpha^a g] \quad \text{(canonical relation, CR).}\end{aligned} \tag{13}$$

The CR can be trivially checked to hold in its algorithmic form in $\bar{\mathcal{B}}(\mathcal{S})$. Elevation of this trivial looking relation in this form to the status of a defining identity of our algebra follows from the composition properties of the Hamilton algebras (HAs) which hold if and only if the CR is assumed to hold. Note that both σ^a and α^a are nonassociative products; the nonassociativity of σ^a is measured by the associator Δ_{σ^a}, a trilinear object, and the rhs of equation (13) can also be written (but for the constant factor a) as an α^a-associator. Thus, the CR is an exact relation between the two associators. Note also that the standard Jordan identity [24] $f^2\sigma^a(g\sigma^a f) = (f^2\sigma^a g)\sigma^a f$ with $f^2 = f\sigma^a f$ follows from (13) by substituting in it $f = h$ and using (9). It is the interaction of the Lie and the Jordan structures via equations (12) and (13) that makes a QHA an interesting algebraic object in its own right.

A classical Hamilton algebra (CHA) $\mathcal{H}^0 = \{\mathcal{H}, \alpha^0, \sigma^0, \mathbb{R}\}$ is now defined by setting $a = 0$ in the identities (9)–(13). Note that the product σ^0 is associative in addition to being commutative—a property of classical phase-space functions. We next turn our attention to the most important characteristic of HAs.

3. Composition properties of Hamilton algebras

Interaction of two quantum systems should result in a composite system describable within the same framework. This intuitive idea is made rigorous by postulating that the tensor-product (TP) composition of two QHAs is yet another QHA. It can be easily verified that the auxiliary product

$$\tau^a = \sigma^a + \sqrt{-a}\alpha^a \tag{14}$$

defined in the complex extension $\mathcal{A}^a$ of $\mathcal{H}^a$, $\tau^a : \mathcal{A}^a \otimes \mathcal{A}^a \to \mathcal{A}^a$ is an associative product. Here the detailed form of the above symbolic relation is

$$f\tau^a g = f\sigma^a g + \sqrt{-a}\, f\alpha^a g \qquad \text{for} \quad f, g \in \mathcal{A}^a.$$

Note that σ^a and α^a are derived products:

$$f\sigma^a g = \tfrac{1}{2}(f\tau^a g + g\tau^a f) \tag{15}$$

$$f\alpha^a g = \frac{1}{2\sqrt{-a}}(f\tau^a g - g\tau^a f). \tag{16}$$

The algebra $\{\mathcal{A}, \tau^a, \mathbb{C}\}^a = \mathcal{A}^a$, where $\mathbb{C}$ is the complex field, is the associative envelope algebra of $\mathcal{H}^a$. We now follow the standard procedure [23] of forming the tensor product $\mathcal{A}^{a_{12}}$ of $\mathcal{A}^{a_1}$ and $\mathcal{A}^{a_2}$. The precise definition of τ_{12} makes use of the 'switching map'

$$S : (\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes (\mathcal{A}_1 \otimes \mathcal{A}_2) \longrightarrow (\mathcal{A}_1 \otimes \mathcal{A}_1) \otimes (\mathcal{A}_2 \otimes \mathcal{A}_2) \tag{17}$$

$$S : (f_1 \otimes f_2) \otimes (g_1 \otimes g_2) \mapsto (f_1 \otimes g_1) \otimes (f_2 \otimes g_2). \tag{18}$$

Then one has

$$\tau_{12} = (\tau_1 \otimes \tau_2) \circ S. \tag{19}$$

Here for brevity we have suppressed the superscripts in the τ. Thus τ_1 stands for $\tau_1^{a_1}$, etc. Similar notation will be used for the other products also. The symbol '$\circ$' denotes the composition of maps. For brevity we shall use the notation $f_{12} \equiv f_1 \otimes f_2$, $(f\sigma g)_{12} \equiv f_{12}\sigma_{12}\, g_{12}$, $[\Delta_\sigma(f, g, h)]_{12} \equiv (f\sigma g)_{12}\,\sigma_{12}\, h_{12} - f_{12}\,\sigma_{12}\,(g\sigma h)_{12}$, etc. Consider two QHAs ${\mathcal{H}_1}^{a_1}$ and ${\mathcal{H}_2}^{a_2}$. Let

$$\tau_{12} = \sigma_{12} + \sqrt{-a_{12}}\alpha_{12} \tag{20}$$

$$\tau_k = \sigma_k + \sqrt{-a_k}\alpha_k \qquad (k = 1, 2). \tag{21}$$

Then equation (19) implies

$$\begin{aligned} \sigma_{12} + \sqrt{-a_{12}}\alpha_{12} &= (\sigma_1 + \sqrt{-a_1}\alpha_1) \otimes (\sigma_2 + \sqrt{-a_2}\alpha_2) \circ S \\ &= [(\sigma_1 \otimes \sigma_2 - \sqrt{a_1 a_2}\alpha_1 \otimes \alpha_2) + (\sqrt{-a_1}\alpha_1 \otimes \sigma_2 + \sqrt{-a_2}\sigma_1 \otimes \alpha_2)] \circ S. \end{aligned} \tag{22}$$

Now equating the symmetric and antisymmetric parts of both sides we obtain the composition properties of the derived products:

$$\sigma_{12} = [(\sigma_1 \otimes \sigma_2) - \sqrt{a_1 a_2}(\alpha_1 \otimes \alpha_2)] \circ S \tag{23}$$

$$\alpha_{12} = \left[\left(\sqrt{\frac{a_1}{a_{12}}}(\alpha_1 \otimes \sigma_2) + \sqrt{\frac{a_2}{a_{12}}}(\sigma_1 \otimes \alpha_2)\right)\right] \circ S. \tag{24}$$

Expanded forms of equations (23) and (24) are

$$(f\sigma g)_{12} = (f\sigma g)_1 \otimes (f\sigma g)_2 - \sqrt{a_1 a_2}(f\alpha g)_1 \otimes (f\alpha g)_2 \tag{25}$$

$$(f\alpha g)_{12} = \sqrt{\frac{a_1}{a_{12}}}(f\alpha g)_1 \otimes (f\sigma g)_2 + \sqrt{\frac{a_2}{a_{12}}}(f\sigma g)_1 \otimes (f\alpha g)_2. \tag{26}$$

With these composition laws of the algebraic products, we now proceed to prove that the identities (9)–(13) also hold in the algebra $\mathcal{H}^{a_{12}}$. This is a *highly nontrivial* result since the TP of two algebras in general leads to an algebra not necessarily of the same type. For example, the TP of two Lie algebras has a product which is symmetric contrary to the antisymmetric nature of the Lie product. Let $f_{12} = f_1 \otimes f_2$, $g_{12} = g_1 \otimes g_2$ and $h_{12} = h_1 \otimes h_2$ denote three arbitrary elements in $\mathcal{H}_1 \otimes \mathcal{H}_2$. For brevity we shall write $f_1 \otimes f_2 = f_1 f_2$, etc. We now demonstrate the following results.

Lemma 1. α_{12} *is antisymmetric.*

Proof.

$$\begin{aligned}(f\alpha g)_{12} &= \sqrt{\frac{a_1}{a_{12}}}(f\alpha g)_1(f\sigma g)_2 + \sqrt{\frac{a_2}{a_{12}}}(f\sigma g)_1(f\alpha g)_2 \\ &= -\sqrt{\frac{a_1}{a_{12}}}(g\alpha f)_1(g\sigma f)_2 - \sqrt{\frac{a_2}{a_{12}}}(g\sigma f)_1(g\alpha f)_2 \\ &= -(g\alpha f)_{12}.\end{aligned} \tag{27}$$

Here in the second line, the antisymmetry of α_k $(k = 1, 2)$ (equation (9)) and the commutativity of σ_k (equation (11)) have been used. □

Proceeding similarly we have

Lemma 2. σ_{12} *is symmetric.*

Lemma 3. α_{12} *satisfies the Jacobi identity*

Proof.

$$\begin{aligned}((f\alpha g)\alpha h)_{12} &= \left[\sqrt{\frac{a_1}{a_{12}}}(f\alpha g)_1(f\sigma g)_2 + \sqrt{\frac{a_2}{a_{12}}}(f\sigma g)_1(f\alpha g)_2\right]\alpha h_{12} \\ &= \sqrt{\frac{a_1}{a_{12}}}\left[\sqrt{\frac{a_1}{a_{12}}}((f\alpha g)\alpha h)_1((f\sigma g)\sigma h)_2 + \sqrt{\frac{a_2}{a_{12}}}((f\alpha g)\sigma h)_1((f\sigma g)\alpha h)_2\right] \\ &\quad + \sqrt{\frac{a_2}{a_{12}}}\left[\sqrt{\frac{a_1}{a_{12}}}((f\sigma g)\alpha h)_1((f\alpha g)\sigma h)_2 + \sqrt{\frac{a_2}{a_{12}}}((f\sigma g)\sigma h)_1((f\alpha g)\alpha h)_2\right] \\ &= -\frac{1}{a_{12}}[(\Delta_\sigma(g,h,f))_1((f\sigma g)\sigma h)_2 + ((f\sigma g)\sigma h)_1(\Delta_\sigma(g,h,f))_2] \\ &\quad + \frac{\sqrt{a_1 a_2}}{a_{12}}((f\alpha g)\sigma h)_1\left\{\sqrt{\frac{a_1}{a_{12}}}((f\alpha h)\sigma g)_2 + \sqrt{\frac{a_2}{a_{12}}}(f\sigma(g\alpha h))_2\right\} \\ &\quad + \frac{\sqrt{a_1 a_2}}{a_{12}}\left\{\sqrt{\frac{a_1}{a_{12}}}((f\alpha h)\sigma g)_1 + \sqrt{\frac{a_2}{a_{12}}}(f\sigma(g\alpha h))_1\right\}((f\alpha g)\sigma h)_2.\end{aligned} \tag{28}$$

Here in the third equality, the CR (13) has been used in the first two terms and the derivation property (12) in the last two terms. Cyclically permuting f, g, h in this relation leads to two similar relations summing which results in separate cancellation of all terms with the prefactor $-1/a_{12}$ and those with the prefactor $\sqrt{a_1 a_2}/a_{12}$, thus proving the Jacobi identity. □

We state two other lemmas the proofs of which are relegated to the appendix because they are lengthy.

Lemma 4. α_{12} *is a derivation wrt* σ_{12}*, i.e. the identity (12) is satisfied in the TP space.*

Lemma 5. σ_{12} *and* α_{12} *are related by the CR (13).*

Now in view of lemmas 1–5, we have

Theorem 1. *The algebra* $\mathcal{H}^{a_{12}}$ *is a QHA.*

This is our main result. This result can be interpreted in the following way. Suppose one starts with two physical systems each describable by its Hamiltonian and its own collection of observables satisfying the evolution given by its Lie bracket α and satisfying the properties expressed by the identities (9)–(13). The two systems may require different PCs for their complete (algebraic) description. Yet, their composite is describable by an 'interaction' Hamiltonian along with its other observables following the evolution by a Lie bracket and also satisfying the same identities (9)–(13) and with a PC which is in principle different from the PCs associated with the components. Thus, two quantum systems with different PCs can in principle interact in a scheme which provides for a consistent dynamics. We shall, however, show that another natural algebraic requirement restricts this possibility further. Implications of this theorem are described in the next section.

Before concluding this section, we note the composition laws of σ and α products for the special case for which the component HAs and their TP are all characterized by the *same* quantum constant a:

$$(f\sigma g)_{12} = (f\sigma g)_1 \otimes (f\sigma g)_2 - a(f\alpha g)_1 \otimes (f\alpha g)_2 \tag{29}$$

$$(f\alpha g)_{12} = (f\alpha g)_1 \otimes (f\sigma g)_2 + (f\sigma g)_1 \otimes (f\alpha g)_2. \tag{30}$$

These laws were earlier derived in [1, 4]. We also note two interesting facts: (a) whereas the composition law of σ alone depends on a, that of α is *independent* of a and (b) the composition law for the α^0 product in a CHA is *identical* to that of the α^a product in a QHA. We now turn to the treatment of mixed quantum and classical HAs.

4. Mixed quantum–classical Hamilton algebra

A hybrid quantum–classical system is of considerable interest from the point of view of quantum measurement theory. In the orthodox Copenhagen philosophy of measurement, in order to measure an observable pertaining to a quantum system, one has to couple a classical system (the apparatus) with it for a certain duration of time during which the measurement takes place and subsequently the latter is decoupled from the former. A measurement is achieved if *unambiguous* information concerning the value of the measured variable is transferred (or 'stored') into some suitable observable of the apparatus and thus one obtains this information (the 'pointer reading') after the decoupling is over. This transfer process is technically referred to as the back-reaction of the quantum system. Theoretically one achieves it by using a coupling Hamiltonian in the TP space of observables of both the systems. It is only natural to look for a Lie product in the TP space which must be constructed out of the Lie products of the component algebras, one of which is the commutator bracket of operators (for the quantum system) and the other, the Poisson bracket of the phase-space functions (for the classical system). Rather than banking upon the *correct* relation (26), one is tempted to be guided by the *inappropriate* relation (30). In this relation, if one considers component 1 as being quantum and component 2 as being classical, one would be naturally led to the mixed bracket (3) with appropriate algorithmic identifications. Note that from our point of view, this would be an *illegal* procedure not permitted in our analysis. Misuse of (30) is the reason why the bracket (3) and equivalently, the bracket (4), do *not* satisfy the Jacobi identity—a fact which has been explicitly checked in [8]. It is the presence of the second term in the rhs of equation (30) which affects the change in the classical variable. This is precisely the term responsible for the back-reaction of system 1 (quantum) on system 2 (classical).

A simple example to illustrate the back-reaction concept can be given for more clarification. Consider two quantum free particle systems labelled $k = 1, 2$ with masses m_k and momenta operators $\hat{p}_k$. Their Hamiltonians are given by $\hat{h}_k = \hat{p}_k^2 \big/ 2m_k$. Let us be interested in measuring $\hat{p}_1$. A convenient coupling Hamiltonian for this purpose is $\hat{h}_{12} = g(t)\hat{p}_1 \otimes \hat{x}_2$ where $\hat{x}_2$ is the position variable of the second particle and $g(t)$ is a coupling parameter such that it is everywhere zero, except between t_0 and $t_0 + \Delta t$ (the duration of measurement), where it is constant ($=g_0$). Confirming to our notation, we have the full Hamiltonian

$$\hat{h}_{12} = \frac{\hat{p}_1^2}{2m_1} \otimes \hat{I}_2 + \hat{I}_1 \otimes \frac{\hat{p}_2^2}{2m_2} + g(t)\hat{p}_1 \otimes \hat{x}_2. \tag{31}$$

Here $\hat{I}_k$ is the unit element of the QHA $\mathcal{H}^k$. Following are the equations of motion, dictated by equation (30):

$$\begin{aligned} \dot{\hat{p}}_1 &= 0 & \dot{\hat{x}}_1 &= \frac{\hat{p}_1}{m_2} + g(t)\hat{x}_2 \\ \dot{\hat{p}}_2 &= -g(t)\hat{p}_1 & \dot{\hat{x}}_2 &= \frac{\hat{p}_2}{m_2}. \end{aligned} \tag{32}$$

Here t is the time parameter and the time derivative is denoted by an overdot. We treat system 1 as one whose momentum $\hat{p}_1$ is to be measured and system 2 as the (quantum) apparatus. The first equation implies that $\hat{p}_1$ does not change as a result of the measurement. On solving for $\hat{p}_2$, we obtain

$$\hat{p}_2 - \hat{p}_2^0 = -g_0\hat{p}_1\Delta t. \tag{33}$$

This equation implies a correlation between $\hat{p}_2 - \hat{p}_2^0$ and $\hat{p}_1$, such that if $\hat{p}_2 - \hat{p}_2^0$ is observed, one can calculate $\hat{p}_1$. This change in momentum of the second system arises due to the back-reaction of the first system. This illustrates how the information transfer between the system and the apparatus takes place in the von Neumann theory [17] of measurement.

We now consider strictly the Bohr point of view that the apparatus has to be a classical system. Suppose that in the above example system 1 is quantum, but system 2 is classical, then it is no longer correct to use equation (30). Instead, one should use relation (26). Let $a_1 = \hbar_1^2 \big/ 4$, $a_2 = 0$ and let $a_{12} = \hbar_{12}^2 \big/ 4$. We then obtain immediately from equations (23) and (24), the results for the mixed products:

$$\sigma_{12} = (\sigma_1 \otimes \sigma_2) \circ S \tag{34}$$

$$\alpha_{12} = \frac{\hbar}{\hbar_{12}}(\alpha_1 \otimes \sigma_2) \circ S. \tag{35}$$

The constant $\hbar_{12}$ being arbitrary, we have to decide on its value from some other consideration. The value $\hbar_{12} = 0$ does not lead to any meaningful structure whereas the choice $\hbar_{12} = \hbar$ would mean that $\mathcal{H}^a \otimes \mathcal{H}^0$ ($a = \hbar^2/4$) is a QHA. It would mean that a mixed quantum–classical system is a *quantum* system governed by a Lie bracket $\alpha_{12} = \left(\alpha_1^a \otimes \sigma_2^0\right) \circ S$. We therefore have

$$(f_1 \otimes f_2)\alpha_{12}(g_1 \otimes g_2) = (f\alpha^a g)_1 \otimes (f\sigma^0 g)_2 \tag{36}$$

$$(f_1 \otimes f_2)\sigma_{12}(g_1 \otimes g_2) = (f\sigma^a g)_1 \otimes (f\sigma^0 g)_2. \tag{37}$$

In algorithmic form, these brackets are

$$[\{Xx, Yy\}]^- = [X, Y]^- xy \tag{38}$$

$$[\{Xx, Yy\}]^+ = [X, Y]^+ xy. \tag{39}$$

Comparison of equations (36) and (30) reveals that the second term in the rhs of (30) is no longer present in the rhs of (36). Following the example of measurement given above it is clear that the absence of the product α_2 in the composition law (36) leads to 'freezing' of classical dynamics. In other words, there is no back-reaction effect of the quantum system on the classical system. This is a *no go* result. Clearly, it does *not* mean that interactions between classical and quantum systems vanish. Interactions exist and are given by elements belonging to the tensor product algebra $\mathcal{H}^a \otimes \mathcal{H}^0$. These, however, do *not* affect any change in the classical 'pointer' variable! This pinpoints the root cause of the impediment to a satisfactory description of dynamical evolution of a mixed quantum–classical system. No wonder that the standard Lie bracket [5, 6, 10] for a mixed quantum–classical system, suggested along the line of equation (30), is *not* compatible with the algebraic requirement. This explains the futility of such approaches. We now turn to another important result.

5. Uniqueness of Planck's constant

In the standard definition [23] of TP of two (associative) algebras $\{A_k, \mu_k, \mathbb{F}\}$ $(k = 1, 2)$, where μ_k is the product law and $\mathbb{F}$ is the field, one has the composition law

$$\mu_{12} = (\mu_1 \otimes \mu_2) \circ S$$

defined in the TP space $A_1 \otimes A_2$. Let e_k be the unit element of A_k, i.e. $(f\mu e)_k = (e\mu f)_k = f_k$. Restriction of the product μ_{12} to the component A_1, denoted as $\mu_{12} \mid A_1$, should result in the product $\mu_1 \otimes e_2$. This means, for two elements $f_1 \otimes e_2, g_1 \otimes e_2 \in A_1 \otimes e_2$, we have

$$\begin{aligned}(f_1 \otimes e_2)\mu_{12}(g_1 \otimes e_2) &= (f\mu g)_1 \otimes (e\mu e)_2 \\ &= (f\mu g)_1 \otimes e_2.\end{aligned} \tag{40}$$

Similarly, $\mu_{12} \mid A_2 = e_1 \otimes \mu_2$.

Extending the above restriction requirement to the HA, we note that α, being a Lie product, does not have a unit element and the Lie product of any element with e (the unit element of the σ product) vanishes, i.e. $f\alpha e = 0$. Algebraically, this means that the HA is *central*, i.e. if $f\alpha x = 0$ holds for arbitrary $f \in \mathcal{H}$, then $x = c\sigma e$ where $c \in \mathbb{R}$. We are now in a position to prove

Theorem 2. *There can be only one PC.*

Proof. Consider the TP $\mathcal{A}^{a_{12}}$ of the associative enveloping algebras of two QHAs $\mathcal{A}^{a_1}$ and $\mathcal{A}^{a_2}$. Then we have

$$\tau_{12} \mid A_1 = \tau_1 \otimes e_2 \qquad \tau_{12} \mid A_2 = e_1 \otimes \tau_2. \tag{41}$$

Using relations (19), (23) and (24) and equating the symmetric and antisymmetric parts separately, we obtain

$$\alpha_{12} \mid A_1 = \sqrt{\frac{a_1}{a_{12}}}(\alpha_1 \otimes e_2) \qquad \alpha_{12} \mid A_2 = \sqrt{\frac{a_2}{a_{12}}}(e_1 \otimes \alpha_2) \tag{42}$$

wherein we have used the relation $(e\sigma e)_k = e_k$. We now invoke the standard restriction requirements:

$$\alpha_{12} \mid A_1 = (\alpha_1 \otimes e_2) \qquad \alpha_{12} \mid A_2 = (e_1 \otimes \alpha_2). \tag{43}$$

Comparison of equations (42) and (43) leads to the result

$$a_{12} = a_1 = a_2 = a \quad \text{(say)} \tag{44}$$

and hence,

$$\hbar_{12} = \hbar_1 = \hbar_2 = \hbar. \tag{45}$$

The result that PC is unique, apart from reaffirming the conventional assumption, also demonstrates clearly the very consistency of the HA approach. □

6. Conclusion

To summarize, we have followed an algebraic approach to QM which is free of the position–momentum generators satisfying the Heisenberg commutation relation. It gives importance to the CR (13) relating both the Lie and the Jordan products. In this way the concept of a QHA subsumes symplectic (i.e. position–momentum generated algebra) as well as nonsymplectic (i.e. algebras for internal degrees of freedom such as spin) variables. We first show that two QHAs with different PCs can lead to a composite QHA with yet another PC. We then show that a QHA can form a composite with a CHA resulting in a QHA and thus allowing for the only consistent description of a mixed mechanics. However, back-reaction of the quantum system on the classical system is shown to be ruled out in this scheme. This result has important bearings on quantum measurement issues as there is no way to describe a quantum system and a (classical) measuring apparatus in a consistent way in the sense of Bohr. We have also proved, based on the natural restriction requirement, that there is but one PC.

In the light of our first result regarding freezing of dynamics in the classical–quantum system interaction, one may wonder how a model such as the one proposed recently by Belavkin [16] is able to deal with the measurement problem. The answer to this puzzle lies in the simple fact that in that model (as in the original von Neumann [17]) the apparatus is also assumed to have a wavefunction (or a wave packet) thus endowing it with quantum property. The model is based on the Schrödinger picture and also one considers only the Lie evolution (dynamics) mediated by stochastic interaction and dissipation. The Jordan product (σ^a in our notation) of observables does *not* enter the treatment of [16]. For example, given two observables f and g, their 'observable' product $f\sigma^a g$ evolves (say, under a Hamiltonian h) according to the product $h\alpha^a(f\sigma^a g)$. So also the observables f and g evolve under the same h. Consistency of time evolution requires the derivation law (12) to hold. Requirements such as this need to be satisfied by the observables of the coupled systems also. It is not possible to make such consistency checks in the Schrödinger picture adopted in [16].

The concept of HA as introduced here pertains to Bose systems. There is a fermionic counterpart. It has been introduced in [2]. In this case one needs a *graded* algebra structure and the identities defining the algebra are graded versions of the identities (9)–(13). Results derived in the present work can be extended to Fermi HAs. This extension along with some results concerning the composition of Bose and Fermi HAs will be dealt with in a separate work.

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Appendix A. Proof of lemma 4

We compute first the lhs of the identity:

$$\begin{aligned}
(f\alpha(g\sigma h))_{12} &= f_{12}\alpha_{12}\{(g\sigma h)_1(g\sigma h)_2 - \sqrt{a_1a_2}(g\alpha h)_1(g\alpha h)_2\}\\
&= \sqrt{\frac{a_1}{a_{12}}}(f\alpha(g\sigma h))_1(f\sigma(g\sigma h))_2 + \sqrt{\frac{a_2}{a_{12}}}(f\sigma(g\sigma h))_1(f\alpha(g\sigma h))_2\\
&\quad - a_1\sqrt{\frac{a_2}{a_{12}}}(f\alpha(g\alpha h))_1(f\sigma(g\alpha h))_2 + a_2\sqrt{\frac{a_1}{a_{12}}}(f\sigma(g\alpha h))_1(f\alpha(g\alpha h))_2\\
&= \sqrt{\frac{a_1}{a_{12}}}((f\alpha g)\sigma h)_1(f\sigma(g\sigma h))_2 + \sqrt{\frac{a_1}{a_{12}}}(g\sigma(f\alpha h))_1(f\sigma(g\sigma h))_2\\
&\quad + \sqrt{\frac{a_2}{a_{12}}}(f\sigma(g\sigma h))_1((f\alpha g)\sigma h)_2 + \sqrt{\frac{a_2}{a_{12}}}(f\sigma(g\sigma h))_1(g\sigma(f\alpha h))_2\\
&\quad - a_1\sqrt{\frac{a_2}{a_{12}}}((f\alpha g)\alpha h)_1(f\sigma(g\alpha h))_2 + a_1\sqrt{\frac{a_2}{a_{12}}}(g\alpha(f\alpha h))_1(f\sigma(g\alpha h))_2\\
&\quad - a_2\sqrt{\frac{a_1}{a_{12}}}(f\sigma(g\alpha h))_1((f\alpha g)\alpha h)_2 + a_2\sqrt{\frac{a_1}{a_{12}}}(f\sigma(g\alpha h))_1(g\alpha(f\alpha h))_2.
\end{aligned} \tag{A1}$$

The two terms on the rhs of the identity are

$$\begin{aligned}
(g\sigma(f\alpha h))_{12} &= g_{12}\sigma_{12}\left[\sqrt{\frac{a_1}{a_{12}}}(f\alpha h)_1(f\sigma h)_2 + \sqrt{\frac{a_2}{a_{12}}}(f\sigma h)_1(f\alpha h)_2\right]\\
&= \sqrt{\frac{a_1}{a_{12}}}(g\sigma(f\alpha h))_1(g\sigma(f\sigma h))_2 + \sqrt{\frac{a_2}{a_{12}}}(g\sigma(f\sigma h))_1(g\sigma(f\alpha h))_2\\
&\quad - a_1\sqrt{\frac{a_2}{a_{12}}}(g\alpha(f\alpha h))_1[((g\alpha f)\sigma h)_2 + (f\sigma(g\alpha h))_2]\\
&\quad - a_2\sqrt{\frac{a_1}{a_{12}}}[((g\alpha f)\sigma h)_1 + (f\sigma(g\alpha h))_1](g\alpha(f\alpha h))_2
\end{aligned} \tag{A2}$$

and

$$\begin{aligned}
((f\alpha g)\sigma h)_{12} &= \left[\sqrt{\frac{a_1}{a_{12}}}(f\alpha g)_1(f\sigma g)_2 + \sqrt{\frac{a_2}{a_{12}}}(f\sigma g)_1(f\alpha g)_2\right]\sigma_{12}h_{12}\\
&= \sqrt{\frac{a_1}{a_{12}}}((f\alpha g)\sigma h)_1((f\sigma g)\sigma h)_2 + \sqrt{\frac{a_2}{a_{12}}}((f\sigma g)\sigma h)_1((f\alpha g)\sigma h)_2\\
&\quad - a_1\sqrt{\frac{a_2}{a_{12}}}((f\alpha g)\alpha h)_1[((f\alpha h)\sigma g)_2 + (f\sigma(g\alpha h))_2]\\
&\quad - a_2\sqrt{\frac{a_1}{a_{12}}}[((f\alpha h)\sigma g)_1 + (f\sigma(g\alpha h))_1]((f\alpha g)\alpha h)_2.
\end{aligned} \tag{A3}$$

We now simplify the combination

$$
\begin{aligned}
&(f\alpha(g\sigma h))_{12} - (g\sigma(f\alpha h))_{12} - ((f\alpha g)\sigma h)_{12} \\
&\quad = -\sqrt{\frac{a_1}{a_{12}}}((f\alpha g)\sigma h)_1(\Delta_\sigma(f,g,h))_2 - \sqrt{\frac{a_2}{a_{12}}}(\Delta_\sigma(f,g,h))_1((f\alpha g)\sigma h)_2 \\
&\qquad + \sqrt{\frac{a_1}{a_{12}}}(g\sigma(f\alpha h))_1(\Delta_\sigma(g,h,f))_2 + \sqrt{\frac{a_2}{a_{12}}}(\Delta_\sigma(g,h,f))_1(g\sigma(f\alpha h))_2 \\
&\qquad + a_1\sqrt{\frac{a_2}{a_{12}}}((f\alpha g)\alpha h)_1((f\alpha h)\sigma g)_2 + a_2\sqrt{\frac{a_1}{a_{12}}}((f\alpha h)\sigma g)_1((f\alpha g)\alpha h)_2 \\
&\qquad + a_1\sqrt{\frac{a_2}{a_{12}}}(g\alpha(f\alpha h))_1((g\alpha f)\sigma h)_2 + a_2\sqrt{\frac{a_1}{a_{12}}}((g\alpha f)\sigma h)_1(g\alpha(f\alpha h))_2.
\end{aligned}
\tag{A4}
$$

Making use of the CR (13) in the associators occurring in the above relation, we immediately see that the rhs vanishes thereby establishing the derivation identity.

Appendix B. Proof of lemma 5

We first simplify the individual terms of the relation.

$$
\begin{aligned}
((f\sigma g)\sigma h)_{12} &= [(f\sigma g)_1(f\sigma g)_2 - \sqrt{a_1a_2}(f\alpha g)_1(f\alpha g)_2]\sigma h_{12} \\
&= ((f\sigma g)\sigma h)_1((f\sigma g)\sigma h)_2 - \sqrt{a_1a_2}((f\sigma g)\alpha h)_1((f\sigma g)\alpha h)_2 \\
&\quad - \sqrt{a_1a_2}((f\alpha g)\sigma h)_1((f\alpha g)\sigma h)_2 + a_1a_2((f\alpha g)\alpha h)_1((f\alpha g)\alpha h)_2 \\
&= ((f\sigma g)\sigma h)_1((f\sigma g)\sigma h)_2 \\
&\quad - \sqrt{a_1a_2}\{((f\alpha h)\sigma g)_1 + (f\sigma(g\alpha h))_1\}\{((f\alpha h)\sigma g)_1 + (f\sigma(g\alpha h))_2\} \\
&\quad - \sqrt{a_1a_2}((f\alpha g)\sigma h)_1((f\alpha g)\sigma h)_2 + (\Delta_\sigma(g,h,f))_1(\Delta_\sigma(g,h,f))_2 \\
&= ((f\sigma g)\sigma h)_1((f\sigma g)\sigma h)_2 - \sqrt{a_1a_2}((f\alpha h)\sigma g)_1((f\alpha h)\sigma g)_2 \\
&\quad - \sqrt{a_1a_2}((f\alpha h)\sigma g)_1(f\sigma(g\alpha h))_2 - \sqrt{a_1a_2}(f\sigma(g\alpha h))_1((f\alpha h)\sigma g)_2 \\
&\quad - \sqrt{a_1a_2}(f\sigma(g\alpha h))_1(f\sigma(g\alpha h))_2 - \sqrt{a_1a_2}((f\alpha g)\sigma h)_1((f\alpha g)\sigma h)_2 \\
&\quad + (\Delta_\sigma(g,h,f))_1(\Delta_\sigma(g,h,f))_2 \\
&= ((f\sigma g)\sigma h)_1((f\sigma g)\sigma h)_2 + ((g\sigma h)\sigma f)_1((g\sigma h)\sigma f)_2 \\
&\quad - ((g\sigma h)\sigma f)_1(g\sigma(h\sigma f))_2 - (g\sigma(h\sigma f))_1((g\sigma h)\sigma f)_2 \\
&\quad + (g\sigma(h\sigma f))_1(g\sigma(h\sigma f))_2 - \sqrt{a_1a_2}((f\alpha h)\sigma g)_1((f\alpha h)\sigma g)_2 \\
&\quad - \sqrt{a_1a_2}((f\alpha h)\sigma g)_1(f\sigma(g\alpha h))_2 - \sqrt{a_1a_2}(f\sigma(g\alpha h))_1((f\alpha h)\sigma g)_2 \\
&\quad - \sqrt{a_1a_2}(f\sigma(g\alpha h))_1(f\sigma(g\alpha h))_2 - \sqrt{a_1a_2}((f\alpha g)\sigma h)_1((f\alpha g)\sigma h)_2.
\end{aligned}
\tag{B1}
$$

Here in the second equality, the derivation relation (12) and the CR (13) have been used and in the fourth equality the associator expressions have been explicitly substituted using (13). Similar simplification steps lead to the result

$$
\begin{aligned}
(f\sigma(g\sigma h))_{12} &= (f\sigma(g\sigma h))_1(f\sigma(g\sigma h))_2 + ((h\sigma f)\sigma g)_1((h\sigma f)\sigma g)_2 \\
&\quad - ((h\sigma f)\sigma g)_1(h\sigma(f\sigma g))_2 - (h\sigma(f\sigma g))_1((h\sigma f)\sigma g)_2 \\
&\quad + (h\sigma(f\sigma g))_1(h\sigma(f\sigma g))_2 - \sqrt{a_1a_2}((f\alpha g)\sigma h)_1((f\alpha g)\sigma h)_2
\end{aligned}
$$

$$- \sqrt{a_1 a_2}((f\alpha g)\sigma h)_1 (g\sigma(f\alpha h))_2 - \sqrt{a_1 a_2}(g\sigma(f\alpha h))_1 ((f\alpha g)\sigma h)_2$$
$$- \sqrt{a_1 a_2}(g\sigma(f\alpha h))_1 (g\sigma(f\alpha h))_2 - \sqrt{a_1 a_2}(f\sigma(g\alpha h))_1 (f\sigma(g\alpha h))_2. \qquad \text{(B2)}$$

The remaining term of the CR is

$$\begin{aligned}
a_{12}((f\alpha h)\alpha g)_{12} \\
&= a_{12}\left[\sqrt{\frac{a_1}{a_{12}}}(f\alpha h)_1(f\sigma h)_2 + \sqrt{\frac{a_1}{a_{12}}}(f\alpha h)_1(f\sigma h)_2\right]\alpha_{12}(g_{12}) \\
&= a_{12}\sqrt{\frac{a_1}{a_{12}}}\left[\sqrt{\frac{a_1}{a_{12}}}((f\alpha h)\alpha g)_1((f\sigma h)\sigma g)_2 + \sqrt{\frac{a_2}{a_{12}}}((f\alpha h)\sigma g)_1((f\sigma h)\alpha g)_2\right] \\
&\quad + a_{12}\sqrt{\frac{a_2}{a_{12}}}\left[\sqrt{\frac{a_1}{a_{12}}}((f\sigma h)\alpha g)_1((f\alpha h)\sigma g)_2 + \sqrt{\frac{a_2}{a_{12}}}((f\sigma h)\sigma g)_1((f\alpha h)\alpha g)_2\right] \\
&= a_1((f\alpha h)\alpha g)_1((f\sigma h)\sigma g)_2 + \sqrt{a_1 a_2}((f\alpha h)\sigma g)_1[((f\alpha g)\sigma h)_2 + (f\sigma(h\alpha g))_2] \\
&\quad + \sqrt{a_1 a_2}[((f\alpha g)\sigma h)_1 + (f\sigma(h\alpha g))_1]((f\alpha h)\sigma g)_2 \\
&\quad + a_2((f\sigma h)\sigma g)_1((f\alpha h)\alpha g)_2 \\
&= (\Delta_\sigma(f, g, h))_1((f\sigma h)\sigma g)_2 + \sqrt{a_1 a_2}((f\alpha h)\sigma g)_1((f\alpha g)\sigma h)_2 \\
&\quad + \sqrt{a_1 a_2}((f\alpha h)\sigma g)_1(f\sigma(h\alpha g))_2 + \sqrt{a_1 a_2}((f\alpha g)\alpha h)_1((f\alpha h)\sigma g)_2 \\
&\quad + \sqrt{a_1 a_2}(f\sigma(h\alpha g))_1((f\alpha h)\sigma g)_2 + ((f\sigma h)\sigma g)_1(\Delta_\sigma(f, g, h))_2 \\
&= ((f\sigma g)\sigma h)_1((f\sigma h)\sigma g)_2 - (f\sigma(g\sigma h))_1((f\sigma h)\sigma g)_2 \\
&\quad + ((f\sigma h)\sigma g)_1((f\sigma g)\sigma h)_2 - ((f\sigma h)\sigma g)_1(f\sigma(g\sigma h))_2 \\
&\quad + \sqrt{a_1 a_2}((f\alpha h)\sigma g)_1((f\alpha g)\sigma h)_2 + \sqrt{a_1 a_2}((f\alpha h)\sigma g)_1(f\sigma(h\alpha g))_2 \\
&\quad + \sqrt{a_1 a_2}((f\alpha g)\sigma h)_1((f\alpha h)\sigma g)_2 + \sqrt{a_1 a_2}(f\sigma(h\alpha g))_1((f\alpha h)\sigma g)_2. \qquad \text{(B3)}
\end{aligned}$$

Combining these three terms leads to the final result

$$((f\sigma g)\sigma h)_{12} - (f\sigma(g\sigma h))_{12} - a_{12}((f\alpha h)\alpha g)_{12} = 0$$

i.e.

$$\Delta_\sigma(f, g, h)_{12} = a_{12}((f\alpha h)\alpha g)_{12}.$$

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